



TITLE:

Global asymptotic stability for a class of difference equations (Theory of Biomathematics and its Applications IV)

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CITATION:

Muroya, Yoshiaki ...[et al]. Global asymptotic stability for a class of difference equations (Theory of Biomathematics and its Applications IV). 数理解析研究所講究録 2008, 1597: 99-105

ISSUE DATE:

2008-05

URL:

<http://hdl.handle.net/2433/81745>

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Global asymptotic stability for a class of difference equations

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1 Introduction

Consider the following nonlinear difference equation with variable coefficients:

$$x_{n+1} = qx_n - \sum_{j=0}^m a_j f_j(x_{n-j}), \quad n = 0, 1, 2, \dots, \quad (1.1)$$

where $0 < q \leq 1$, $a_j \geq 0$, $0 \leq j \leq m$ and $\sum_{j=0}^m a_j > 0$. We now assume that

$$\begin{cases} f(x) \in C(-\infty, +\infty) \text{ is a strictly monotone increasing function,} \\ f(0) = 0, \quad 0 < \frac{f_j(x)}{f(x)} \leq 1, \quad x \neq 0, \quad 1 \leq j \leq m, \quad \text{and} \\ \text{if } f(x) \neq x, \text{ then } \lim_{x \rightarrow -\infty} f(x) \text{ is finite, otherwise } f(x) = x. \end{cases} \quad (1.2)$$

The above difference equation has been studied by many literatures (see for example, [1]-[9] and references therein).

Definition 1.1 The solution y^* of (1.1) is called *uniformly stable*, if for any $\epsilon > 0$ and non-negative integer n_0 , there is a constant $\delta = \delta(\epsilon) > 0$ such that $\sup\{|y_{n_0-i} - y^*| \mid 0 \leq i \leq m\} < \delta$, implies that the solution $\{y_n\}_{n=0}^\infty$ of (1.1) satisfies $|y_n - y^*| < \epsilon$, $n = n_0, n_0 + 1, \dots$.

Definition 1.2 The solution y^* of (1.1) is called *globally attractive*, if every solution of (1.1) tends to y^* as $n \rightarrow \infty$.

Definition 1.3 The solution y^* of (1.1) is called *globally asymptotically stable*, if it is uniformly stable and globally attractive.

In this paper, we study "semi-contractive" functions and global asymptotic stability of difference equations. In Section 2, we first define semi-contractivity of functions and show the related results on the global asymptotic stability of difference equations.

2 Semi-contractive function

Assume that

$$g(z_0, z_1, \dots, z_m) \in C(R^{m+1}) \quad \text{and} \quad g(y, y, \dots, y) = y \text{ has a unique solution } y = y^*. \quad (2.1)$$

Definition 2.1 The function $g(z_0, z_1, \dots, z_m)$ is said to be semi-contractive at y^* , if

- (i) for any constants $\underline{z} < y^*$ and $z_i \geq \underline{z}$, $0 \leq i \leq m$, there exists a constant $y^* < \bar{z} < +\infty$ such that $g(z_0, z_1, \dots, z_m) \leq \bar{z}$, and for any $\underline{z} \leq z_i \leq \bar{z}$, $0 \leq i \leq m$, there exists a constant $\tilde{z} > \underline{z}$ such that $\tilde{z} \leq g(z_0, z_1, \dots, z_m)$, or
- (ii) for any constants $\bar{z} > y^*$ and $z_i \leq \bar{z}$, $0 \leq i \leq m$, there exists a constant $y^* > \underline{z} > -\infty$ such that $g(z_0, z_1, \dots, z_m) \geq \underline{z}$, and for any $\underline{z} \leq z_i \leq \bar{z}$, $0 \leq i \leq m$, there exists a constant $\tilde{z} < \bar{z}$ such that $\tilde{z} \geq g(z_0, z_1, \dots, z_m)$.

Lemma 2.1 If $g(y) \in C(R)$ is a strictly monotone decreasing function such that $g(g(y)) > y$ for any $y < y^*$, then $g(z)$ is semi-contractive for y^* .

Lemma 2.2 Assume (2.1) and that each $g_i(z_0, z_1, \dots, z_m)$, $0 \leq i \leq m$ is semi-contractive for y^* . Then for any $b_{n,i} \geq 0$, $n \geq 0$, $0 \leq i \leq m$ such that $\sum_{i=0}^m b_{n,i} = 1$ and $\lim_{n \rightarrow \infty} b_{n,i} = b_i$, $0 \leq i \leq m$, it holds that $\sum_{i=0}^m b_{n,i} g_i(z_0, z_1, \dots, z_m)$ is semi-contractive for y^* .

Corollary 2.1 Assume (2.1) and that $g(z_0, z_1, \dots, z_m)$ is semi-contractive for y^* . Then for any $0 \leq q_n < 1$, $g_n(z_0, z_1, \dots, z_m)$ and k such that

$$\begin{cases} \lim_{n \rightarrow \infty} q_n = q < 1, & \text{and } 0 \leq k \leq m, \\ \lim_{n \rightarrow \infty} g_n(z_0, z_1, \dots, z_m) = g(z_0, z_1, \dots, z_m) & \text{for any } z_0, z_1, \dots, z_m \in (-\infty, +\infty), \end{cases} \quad (2.2)$$

it holds that $q_n z_k + (1 - q_n) g_n(z_0, z_1, \dots, z_m)$ is semi-contractive for y^* .

Corollary 2.2 Assume that each $g_i(z) \in C(R)$ and $g_i(y) = y$ has a unique solution $y = y^*$, $0 \leq i \leq m$, and each $g_i(z_i)$, $0 \leq i \leq m$ is semi-contractive for y^* , then for any $b_{n,i} \geq 0$, $n \geq 0$, $0 \leq i \leq m$ such that $\sum_{i=0}^m b_{n,i} = 1$ and $\lim_{n \rightarrow \infty} b_{n,i} = b_i$, $0 \leq i \leq m$, it holds that $\sum_{i=0}^m b_{n,i} g_i(z_i)$ is semi-contractive for y^* . In particular, for any $0 \leq q_n < 1$ and k such that $\lim_{n \rightarrow \infty} q_n = q < 1$ and $0 \leq k \leq m$, it holds that $q_n z_k + (1 - q_n) \sum_{i=0}^m b_{n,i} g_i(z_i)$ is semi-contractive for y^* .

Remark 2.1 If $g(z_0, z_1, \dots, z_m) > 0$ for any $z_i > 0$, $0 \leq i \leq m$, then there are cases that we may restrict our attention only to $z_i > 0$, $0 \leq i \leq m$ and the unique positive solution $y^* > 0$ of $g(y^*, y^*, \dots, y^*) = y^*$, whether or not $g(y, y, \dots, y) = y$ has other solutions $y \leq 0$.

Example 2.1 Examples of semi-contractive function $g(z_0, z_1, \dots, z_m)$ for y^* .

- (i) $g(z_0, z_1, \dots, z_m) = z_m e^{c(1-z_m)}$, $y^* = 1$ and $c \leq 2$ (see [1]).
- (ii) $g(z_0, z_1, \dots, z_m) = z_0 \exp\{c(1 - \sum_{i=0}^m a_i z_i)\}$, $y^* = 1/(\sum_{i=0}^m a_i)$ and $c \leq 2$, where $a_0 > 0$, $a_i \geq 0$, $1 \leq i \leq m$ and $(\sum_{i=1}^m a_i)/a_0 \leq 2/e$.
This is equivalent that $h(u_0, u_1, \dots, u_m) = u_0 - c \sum_{i=0}^m b_i (e^{u_i} - 1)$ is semi-contractive for $u^* = 0$ and $c \leq 2$, where $z_i = y^* e^{u_i}$, $b_0 = y^* a_0 > 0$, $b_i = y^* a_i \geq 0$, $1 \leq i \leq m$, $\sum_{i=0}^m b_i = 1$, and $(\sum_{i=1}^m b_i)/b_0 \leq 2/e$ (see [8]).
- (iii) $g(z_0, z_1, \dots, z_m) = c(1 - e^{z_m})$, $y^* = 0$ and $c \leq 1$ (see [3]).
- (iv) $g(z_0, z_1, \dots, z_m) = \frac{cz_m}{1+bz_m^p}$, $x^* = ((c-1)/b)^{1/p}$ and $c \leq \frac{p}{p-2}$, where $p > 2$ and $b > 0$ (see [1]).

We consider the following difference equation

$$y_{n+1} = q_n y_{n-k} + (1 - q_n) g_n(y_n, y_{n-1}, \dots, y_{n-m}), \quad n = 0, 1, \dots, \quad (2.3)$$

where we assume (2.1) and

$$\begin{cases} 0 \leq q_n < 1, \quad \lim_{n \rightarrow \infty} q_n = q < 1, \quad k \in \{0, 1, \dots, m\}, \quad \text{and} \\ \lim_{n \rightarrow \infty} g_n(z_0, z_1, \dots, z_m) = g(z_0, z_1, \dots, z_m) \quad \text{for any } z_0, z_1, \dots, z_m \in (-\infty, +\infty). \end{cases} \quad (2.4)$$

Theorem 2.1 *If $g(z_0, z_1, \dots, z_m)$ is semi-contractive for y^* , then y^* of (2.3) is globally asymptotically stable for any $0 \leq q < 1$.*

Collorary 2.3 *Assume that there exists a constant $0 \leq q_0 < 1$ and some $0 \leq k \leq m$ such that $q_0 z_k + (1 - q_0)g(z_0, z_1, \dots, z_m)$, is semi-contractive for y^* . Then, for any $q_0 \leq q_n < 1$ and $g_n(z_0, z_1, \dots, z_m)$ which satisfy (2.4), the solution y^* of (2.3) is globally asymptotically stable.*

Remark 2.2 (i) The corresponding continuous case (2.3) is the following differential equation

$$\begin{cases} y'(t) = -p(t)\{y(t) - \frac{1}{1-q_n}g_n(y(n), y(n-1), \dots, y(n-m))\}, \quad n \leq t < n+1, \quad n = 0, 1, 2, \dots, \\ p(t) > 0, \quad q_n = e^{-\int_n^{n+1} p(t)dt} < 1. \end{cases}$$

(ii) In Theorem 2.1, a semi-contractivity condition is a delays and q_n -independent condition for the solution y^* of (2.3) to be globally asymptotically stable.

By Theorem 2.1 and Example 2.1, we obtain the following result:

Example 2.2 Examples of delays and q -independent stability conditions.

(i) Ricker model $y_{n+1} = qy_n + (1-q)y_{n-m}e^{c(1-y_{n-m})}$, $n = 0, 1, 2, \dots$. The positive equilibrium $y^* = 1$ is globally asymptotically stable, if $c \leq 2$ (see [1]).

(ii) Ricker model with delayed-density dependence $y_{n+1} = qy_n + (1-q)y_n \exp\{c(1 - \sum_{i=0}^m a_i y_{n-i})\}$. The positive equilibrium $y^* = 1/(\sum_{i=0}^m a_i)$ is globally asymptotically stable, if $c \leq 2$, where $a_0 > 0$, $a_i \geq 0$, $1 \leq i \leq m$ and $(\sum_{i=1}^m a_i)/a_0 \leq 2/e$ (see [8]).

(iii) Wazewska-Czyzewska and Lasota model $y_{n+1} = qy_n + (1-q)c \sum_{i=0}^m b_i e^{-\gamma y_{n-i}}$, $n = 0, 1, 2, \dots$,

where $\gamma > 0$, $b_i \geq 0$, $0 \leq i \leq m$, and $\sum_{i=0}^m b_i = 1$.

The positive equilibrium y^* is the positive solution of the equation $y^* = ce^{-\gamma y^*}$. Put $x_n = \gamma(y^* - y_n)$. Then, this equation is equivalent to

$$x_{n+1} = qx_n - (1-q)\gamma y^* \sum_{i=0}^m b_i (e^{x_{n-i}} - 1), \quad \text{where } b_i \geq 0, \quad 0 \leq i \leq m, \quad \sum_{i=0}^m b_i = 1. \quad (2.5)$$

Thus, the positive equilibrium y^* is globally asymptotically stable, if $c \leq e/\gamma$ which is equivalent that the zero solution of (2.5) is globally asymptotically stable if $\gamma y^* \leq 1$ (see [3]).

(iv) Bobwhite quail population model $y_{n+1} = qy_n + (1-q)\frac{cy_{n-m}}{1+by_{n-m}^p}$, $n = 0, 1, 2, \dots$, where $c > 1$, $b > 0$. The positive equilibrium $y^* = ((c-1)/b)^{1/p}$ is globally asymptotically stable, if $c \leq \frac{p}{p-2}$ for $p > 2$ (see [1]).

We have the following counter example:

Example 2.3 Examples of q -dependent and delay-dependent stability conditions.

(i) A model in hematopoiesis $y_{n+1} = qy_n + (1-q)e^{2(1-y_n)}$, $n = 0, 1, 2, \dots$.

The equilibrium $y^* = 1$ is globally asymptotically stable if $q \in [1/3, 1)$, and 2-cycle if $q \in [0, 1/3)$ (see [2]).

(ii) A delayed model in hematopoiesis $y_{n+1} = qy_n + (1-q)e^{2(1-y_{n-2})}$, $n = 0, 1, 2, \dots$.

The characteristic equation takes the form $\lambda^3 - q\lambda^2 = -2(1-q)$. Then for $q = q_2 = \frac{3-\sqrt{3}}{2} =$

$0.633975 \dots > 1/3$, the roots are $-1 < \lambda_1 < 0$, $|\lambda_2| = |\lambda_3| = 1$. For $q_2 < q < 1$, the equilibrium $y^* = 1$ is locally attractive but it becomes unstable for $q = q_2$, and *Hopf bifurcation occurs* (see [2]).

(iii) Ricker's equation with delayed-density dependence $y_{n+1} = y_n \exp\{c_n(1 - \sum_{i=0}^m b_{n,i} y_{n-i})\}$, $n = 0, 1, \dots$, which is equivalent to $x_{n+1} = x_n - c_n \sum_{i=0}^m b_{n,i} (e^{x_{n-i}} - 1)$, $n = 0, 1, \dots$, where $c_n, b_{n,i} > 0$, $\sum_{i=0}^m b_{n,i} = 1$ and $y_n = e^{x_n}$.

The positive equilibrium $y^* = 1$ is *globally asymptotically stable* if $\limsup_{n \rightarrow \infty} \sum_{i=n}^{n+m} r_i < \frac{3}{2} + \frac{1}{2(m+1)}$ (see [7]).

(iv) A model of the growth of bobwhite quail populations $y_{n+1} = qy_n + (1-q)\frac{cy_n}{1+y_{n-m}^p}$, $n = 0, 1, \dots$,

where $c, p > 0$. If $c \leq 1$, then for any $0 < q < 1$, $\lim_{n \rightarrow \infty} y_n = 0$. If $c > 1$, then the positive equilibrium $y^* = (c-1)^{1/p}$ of the model exists. Moreover, if $p \leq \frac{2c}{(c-1)(1-q)}$ for $m = 0$, or $p < \frac{c}{(c-1)(1-q)} \frac{3m+4}{2(m+1)^2}$ for $m \geq 1$, then the positive equilibrium y^* is globally asymptotically stable (see [4]).

3 Delays-independent stability conditions for (1.1)

After setting

$$r_1 = a_0, r_2 = \sum_{i=1}^m a_i, r = r_1 + r_2, \varphi(x) = qx - r_1 f(x), \hat{z}(q) = (-1 + \sqrt{1+4q})/(2q), \quad (3.1)$$

we have the following result.

Theorem 3.1 Assume that $f(x) = f_0(x) = e^x - 1$ and $0 < q < 1$, and suppose that

$$r_1 < q, \quad r \leq q + (1-q) \ln(q/r_1) \quad \text{and} \quad (q/r_1)^q e^{r-q} (r_1 - r_2) + (1-q) \geq 0, \quad (3.2)$$

or

$$\begin{cases} r_1 \leq q, & r > q + (1-q) \ln(q/r_1), & qr_2 \leq r_1, \\ r - r_2(q/r_1)^q e^{r-q} - (1-q)(\bar{L} - 1) \geq 0 & \text{and} & \bar{L} = \ln \frac{r-q-(1-q)\ln(q/r_1)}{r_2} \leq 0, \end{cases} \quad (3.3)$$

or

$$\begin{cases} r_1 > q, & r \leq 1+q, & r - r_2(q/r_1)^q e^{r-q} - (1-q)(\ln(q/r_1) - 1) \geq 0, \\ \text{and} & \frac{r}{q}(q/r_1)^q e^{r-q} \leq \frac{e^{\hat{z}(q)}}{1-\hat{z}(q)}. \end{cases} \quad (3.4)$$

Then, the zero solution of (1.1) is globally asymptotically stable.

Numerical result 3.1 Assume that $f(x) = f_0(x) = e^x - 1$ and $0 < q < 1$.

(i) The last inequality in (3.4) can be eliminated from (3.4).

(ii) Under the condition $\frac{r_2}{r_1} \leq \frac{2}{e}$ and $r \leq 1+q$, the third inequality of (3.4) is satisfied, and hence the zero solution of (1.1) is globally asymptotically stable.

Example 3.1 Wazewska-Czyżewska and Lasota model (see [9]).

$$y_{n+1} = qy_n + (1-q)c \sum_{i=0}^m b_i e^{-\gamma y_{n-i}}, \quad \text{where } c, \gamma > 0, b_i \geq 0 \text{ and } \sum_{i=0}^m b_i = 1. \quad (3.5)$$

(3.5) is equivalent to (2.5). For equation (3.5), the positive equilibrium of (3.5), say y^* , is globally asymptotically stable, if $\gamma y^* \leq 1$ (see [3] and Example 2.2 iii)). For the case $\gamma y^* > 1$, by using

the generalized Yorke condition, [6, Theorem 8] extended these to $\gamma y^* \leq (1 + q^{m+1})/(1 - q^{m+1})$ with some restricted conditions " $V_k(q) < 0$, $W_k(q) < 0$ ". Note that the last condition contains the restriction $(q + q^2 + \dots + q^m)q^m \leq 1$ for $0 < q < 1$. On the other hand, by applying Theorem 3.1 and Numerical result 3.1 to (2.5) for $a_i = (1 - q)\gamma y^* b_i$, $0 \leq i \leq m$, we obtain another sufficient condition, for example, $\sum_{i=1}^m b_i \leq \frac{2}{e} b_0$ and $\gamma y^* \leq (1 + q)/(1 - q)$ for the solution y^* of (3.5) to be globally asymptotically stable. Note that $e^x - 1 < x/(1 - x)$ for $0 < x < 1$ and $\frac{1+q^{m+1}}{1-q^{m+1}} < \frac{1+q}{1-q}$ for $0 < q < 1$. Thus, compared with [6, Proof of Theorem 2] (and [1]-[9] and references therein), one can see that our results offer new stability conditions to (3.5).

4 Semi-contractivity with a sign condition

For $0 \leq q < 1$, consider the following nonautonomous equation

$$x_{n+1} = qx_n - \sum_{j=0}^m a_{n,j} f_j(x_{n-j}), \quad n = 0, 1, \dots, \quad (4.1)$$

where $0 < q \leq 1$, $a_{n,j} \geq 0$, $0 \leq j \leq m$, $n = 0, 1, \dots$, and $\sum_{j=0}^m a_{n,j} > 0$, and we assume that there is a function $f(x)$ such that (1.2) holds.

For (4.1) and any $0 \leq l_n \leq m$, we can derive the following equation.

$$\begin{cases} x_{n+1} = \{q^{l_n+1} x_{n-l_n} + (1-q) \sum_{k=0}^{l_n} q^k \sum_{j=0}^{m-k} a_{n-k,j} f_j(x_{n-k-j})\} \\ \quad - \sum_{k=1}^{l_n} q^k \sum_{j=m-k+1}^m a_{n-k,j} f_j(x_{n-k-j}), \quad n = 2m, 2m+1, \dots \end{cases} \quad (4.2)$$

Similar to the proofs of [5, Lemmas 2.3 and 2.4], we have the following two lemmas for (4.1).

Lemma 4.1 *Let $\{x_n\}_{n=0}^\infty$ be the solution of (4.1). If there exists an integer $n \geq m$ such that $x_{n+1} \geq 0$ and $x_{n+1} > x_n$, then there exists an integer $\underline{g}_n \in [n - m, n]$ such that*

$$x_{\underline{g}_n} = \min_{0 \leq j \leq m} x_{n-j} < 0. \quad (4.3)$$

If there exists an integer $n \geq m$ such that $x_{n+1} \leq 0$ and $x_{n+1} < x_n$, then there exists an integer $\bar{g}_n \in [n - m, n]$ such that

$$x_{\bar{g}_n} = \max_{0 \leq j \leq m} x_{n-j} > 0. \quad (4.4)$$

After setting

$$\begin{cases} \bar{r}_1 = \sup_{n \geq m} \sum_{k=0}^m q^k \sum_{j=0}^{m-k} a_{n-k,j}, & \bar{r}_2 = \sup_{n \geq m} \sum_{k=1}^m q^k \sum_{j=m-k+1}^m a_{n-k,j}, \\ \bar{r} = \bar{r}_1 + \bar{r}_2, & \bar{\varphi}(x) = \bar{q}x - \bar{r}_1 f(x), \quad \bar{q} = q^{m+1}, \quad \bar{z} = (-1 + \sqrt{1 + 4\bar{q}})/(2\bar{q}), \end{cases} \quad (4.5)$$

and

$$\bar{g}(z_0, z_1, \dots, z_m; \bar{q}) = \bar{\varphi}(z_0) + \sum_{k=1}^m q^k \sum_{j=m-k+1}^m a_{n-k,j} g(z_j), \quad (4.6)$$

we are able to prove the following results.

If there exists an integer $n \geq m$ such that $x_{n+1} \geq 0$ and $x_{n+1} > x_n$, then by (4.3) and (4.2) with $l_n = n - g_n$, we have that

$$x_{n+1} \leq \bar{\varphi}(x_{g_n}) - \bar{r}_2 f(L_n), \quad L_n = \min_{0 \leq j \leq 2m} x_{n-j}. \quad (4.7)$$

If there exists an integer $n \geq m$ such that $x_{n+1} \leq 0$ and $x_{n+1} < x_n$, then by (4.4) and (4.2) with $l_n = n - \bar{g}_n$, we have that

$$x_{n+1} \geq \bar{\varphi}(x_{\bar{g}_n}) - \bar{r}_2 f(R_n), \quad R_n = \max_{0 \leq j \leq 2m} x_{n-j}. \quad (4.8)$$

Lemma 4.2 Suppose that the solution x_n of (4.1) is oscillatory about 0. If for some real number $L < 0$, there exists a positive integer $n_L \geq 2m$ such that $x_n \geq L$ for $n \geq n_L$, then for any integer $n \geq n_L + 2m$,

$$x_{n+1} \leq R_L \text{ for } n \geq n_L + 2m, \quad \text{and} \quad x_{n+1} \geq S_L \text{ for } n \geq n_L + 4m, \quad (4.9)$$

where $R_L = \max_{L \leq x \leq 0} \varphi(x) - r_2 f(L) > 0$ and $S_L = \min_{0 \leq x \leq R_L} \varphi(x) - r_2 f(R_L) < 0$. Moreover, if $S_L > L$ for any $L < 0$, then $\lim_{n \rightarrow \infty} x_n = 0$.

Assume that $g(z_0, z_1, \dots, z_m)$ is continuous for $(z_0, z_1, \dots, z_m) \in R^{m+1}$ and $g(y^*, y^*, \dots, y^*) = y^*$ has a unique solution y^* .

Definition 4.1 The function $g(z_0, z_1, \dots, z_m)$ is said to be semi-contractive with a sign condition z_0 for y^* , if

(i) for any constants $\underline{z} < y^*$ and $z_i \geq \underline{z}$, $0 \leq i \leq m$ with $z_0 \leq y^*$, there exists a constant $y^* < \bar{z} < +\infty$ such that $g(z_0, z_1, \dots, z_m) \leq \bar{z}$ and for any $\underline{z} \leq z_i \leq \bar{z}$, $0 \leq i \leq m$ with $z_0 \geq y^*$, there exists a constant $\bar{z} > \underline{z}$ such that $\bar{z} \leq g(z_0, z_1, \dots, z_m)$,

or

(ii) for any constants $\bar{z} > y^*$ and $z_i \leq \bar{z}$, $0 \leq i \leq m$ with $z_0 \geq y^*$, there exists a constant $y^* > \underline{z} > -\infty$ such that $g(z_0, z_1, \dots, z_m) \geq \underline{z}$ and for any $\underline{z} \leq z_i \leq \bar{z}$, $0 \leq i \leq m$ with $z_0 \leq y^*$, there exists a constant $\bar{z} < \underline{z}$ such that $\bar{z} \geq g(z_0, z_1, \dots, z_m)$.

Then by (4.7), (4.8) and Lemma 4.2, we can obtain the following result.

Theorem 4.1 If $\bar{g}(z_0, z_1; \bar{q}) = \bar{\varphi}(z_0) - \bar{r}_2 f(z_1)$ is semi-contractive with a sign condition z_0 for $x^* = 0$, then the zero solution of (4.1) is globally asymptotically stable.

Note that if $\bar{g}(z_0, z_1; \bar{q}) = \bar{\varphi}(z_0) - \bar{r}_2 f(z_1)$ is semi-contractive with a sign condition z_0 for $x^* = 0$, then the zero solution $x^* = 0$ of (4.1) is uniformly stable and hence $x^* = 0$ is globally asymptotically stable.

For the special case $f(x) = e^x - 1$, we establish the following sufficient conditions for $0 < q < 1$ which are some extensions of the result in [5] for $q = 1$.

Theorem 4.2 Suppose that $f(x) = e^x - 1$ and that one of the following condition is fulfilled:

$$\begin{cases} \bar{r}_2 \leq 1 \quad \text{and} \quad \frac{\bar{r}}{\bar{q}} e^{\bar{r}_2} \leq \frac{e^{\bar{z}}}{1-\bar{z}} & \text{if } \bar{r}_1 \leq \bar{q}, \\ \bar{r} \leq 1 + \bar{q} \quad \text{and} \quad \frac{\bar{r}}{\bar{q}} (\bar{q}/\bar{r}_1)^{\bar{q}} e^{\bar{r}-\bar{q}} \leq \frac{e^{\bar{z}}}{1-\bar{z}} & \text{if } \bar{r}_1 > \bar{q}, \end{cases} \quad (4.10)$$

$$\text{or} \quad \begin{cases} \bar{r}_2 \leq 1, \quad \frac{\bar{r}}{\bar{q}} e^{\bar{r}_2} > \frac{e^{\bar{z}}}{1-\bar{z}} \quad \text{and} \quad G_3(\delta) > 0 & \text{if } \bar{r}_1 \leq \bar{q}, \\ \bar{r} \leq 1 + \bar{q}, \quad \frac{\bar{r}}{\bar{q}} (\bar{q}/\bar{r}_1)^{\bar{q}} e^{\bar{r}-\bar{q}} > \frac{e^{\bar{z}}}{1-\bar{z}} \quad \text{and} \quad G_1(\alpha) > 0 & \text{if } \bar{r}_1 > \bar{q}, \end{cases} \quad (4.11)$$

$$\text{with } \begin{cases} G_1(x) = \bar{q} \left(\bar{q} \ln(\bar{q}/\bar{r}_1) + \bar{r} - \bar{q} - \bar{r}_2 e^x \right) + \bar{r} - \bar{r}(\bar{q}/\bar{r}_1)^{\bar{q}} e^{\bar{r}-\bar{q}-\bar{r}_2 e^x} - x, \\ G_3(x) = (\bar{r}_1 + (1 + \bar{q})\bar{r}_2) - \bar{q}\bar{r}_2 e^x - \bar{r}e^{\bar{r}_2 - \bar{r}_2 e^x} - x, \end{cases} \quad (4.12)$$

where α and δ are the lowest solutions of $G_1(x) = 0$ and $G_3(x) = 0$, respectively, and \bar{z} is a positive solution of $\bar{q}z^2 + z - 1 = 0$. Then, the solution $x^* = 0$ of (4.1) is globally asymptotically stable.

As an immediate consequence we have the following corollary.

Corollary 4.1 Assume that $f(x) = e^x - 1$ and that

$$\bar{r} \leq 1 + \bar{q} \quad \text{and} \quad \bar{r}_1 \geq \bar{q}\bar{r}_2. \quad (4.13)$$

If

$$(i) \frac{\bar{r}}{\bar{q}} (\bar{q}/\bar{r}_1)^{\bar{q}} e^{\bar{r}-\bar{q}} \leq \frac{e^{\bar{z}}}{1-\bar{z}}, \quad \text{or} \quad (ii) \frac{\bar{r}}{\bar{q}} (\bar{q}/\bar{r}_1)^{\bar{q}} e^{\bar{r}-\bar{q}} > \frac{e^{\bar{z}}}{1-\bar{z}} \quad \text{and} \quad G_1(\alpha) > 0, \quad (4.14)$$

then, the zero solution of (4.1) is globally asymptotically stable.

Example 4.1 Consider a model $x_{n+1} = qx_n - \sum_{i=0}^m a_i(e^{-x_{n-i}} - 1)$, $n = 0, 1, 2, \dots$, where $a_i \geq 0$, $0 \leq i \leq m$, and $\sum_{i=0}^m a_i > 0$. This equation is equivalent to (2.5), if $\sum_{i=0}^m a_i = (1-q)\gamma y^*$ and $0 < q < 1$. By Corollary 4.1, the zero solution $x^* = 0$ is globally asymptotically stable for $\bar{r} \leq 1 + \bar{q}$, if for the setting (4.5) and $\hat{r}_1 = \bar{q}(\frac{1+\bar{q}}{\bar{q}}(1-\bar{z})e^{1-\bar{z}})^{1/\bar{q}}$, it holds that $\frac{\bar{r}_2}{\bar{r}_1} \leq \frac{1+\bar{q}}{\bar{r}_1} - 1$. Since $e^x - 1 < x/(1-x)$ for $0 < x < 1$ and we do not need the restriction $(q + q^2 + \dots + q^m)q^m \leq 1$ for $0 < q < 1$ in [6, Theorem 2], our results improve some of [6, Theorem 8] (see [5]).

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